CHARACTERIZATION OF ISOMETRIC EMBEDDINGS OF GRASSMANN GRAPHS

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Abstract. Let V be an n-dimensional left vector space over a division ring R. We write $\mathcal{G}_k(V)$ for the Grassmannian formed by k-dimensional subspaces of V and denote by $\Gamma_k(V)$ the associated Grassmann graph. Let also V' be an n'-dimensional left vector space over a division ring R'. Isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ are classified in [13]. A classification of $J(n,k)$-subsets in $\mathcal{G}_{k'}(V')$, i.e. the images of isometric embeddings of the Johnson graph $J(n,k)$ in $\Gamma_{k'}(V')$, is presented in [12]. We characterize isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ as mappings which transfer apartments of $\mathcal{G}_k(V)$ to $J(n,k)$-subsets of $\mathcal{G}_{k'}(V')$. This is a generalization of the earlier result concerning apartments preserving mappings [11, Theorem 3.10].

1. Introduction

Let V be an n-dimensional left vector space over a division ring R and let $\mathcal{G}_k(V)$ be the Grassmannian formed by k-dimensional subspaces of V. The associated Grassmann graph will be denoted by $\Gamma_k(V)$. By classical Chow's theorem [2], every automorphism of $\Gamma_k(V)$ with $1 < k < n-1$ is induced by a semilinear automorphism of V or a semilinear isomorphism of V to the dual vector space V^* and the second possibility can be realized only in the case when $n = 2k$. The statement fails for $k = 1, n-1$. In this case, any two distinct vertices of $\Gamma_k(V)$ are adjacent and any bijective transformation of $\mathcal{G}_k(V)$ is an automorphism of $\Gamma_k(V)$.

Results closely related to Chow's theorem can be found in [1, 3, 4, 6, 7, 9, 10], see also [11, Section 3.2].

One of resent generalizations of Chow's theorem is the classification of isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$, where V' is an n'-dimensional left vector space over a division ring R' [13]. The existence of such embeddings implies that

$$\min\{k, n-k\} \leq \min\{k', n'-k'\}, \tag{1.1}$$

i.e. the diameter of $\Gamma_k(V)$ is not greater than the diameter of $\Gamma_{k'}(V')$. The case $k = 1, n-1$ is trivial: every isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ is a bijection to a clique of $\Gamma_{k'}(V')$. If $1 < k < n-1$ then isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ are defined by semilinear $(2k)$-embeddings, i.e. semilinear injections which transfer any $2k$ linearly independent vectors to linearly independent vectors.

A result of similar nature is obtained in [12]. This is the classification of the images of isometric embeddings of the Johnson graph $J(n,k)$ in the Grassmann graph $\Gamma_{k'}(V')$. As above, we need (1.1) which guarantees that the diameter of $J(n,k)$ is not greater than the diameter of $\Gamma_{k'}(V')$. The images of isometric embeddings of $J(n,k)$ in $\Gamma_{k'}(V')$ will be called $J(n,k)$-*subsets* of $\mathcal{G}_{k'}(V')$.

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Suppose that $1 < k < n-1$ (the case $k = 1, n-1$ is trivial). If $n = 2k$ then every $J(n,k)$-subset is an apartment in a parabolic subspace of $\mathcal{G}_{k'}(V')$ and we get an apartment of $\mathcal{G}_{k'}(V')$ if $n = n'$ and $k' = k, n-k$. In the case when $n \neq 2k$, there are two distinct types of $J(n,k)$-subsets.

If $n = n'$ then every apartments preserving mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_k(V')$ with $1 < k < n-1$ is induced by a semilinear embedding of V in V' or a semilinear embedding of V in V'^* and the second possibility can be realized only in the case when $n = 2k$ [11, Theorem 3.10]. For $k = 1, n-1$ this fails. By [5], there are apartments preserving mappings of $\mathcal{G}_1(V)$ to itself which can not be defined by semilinear mappings.

Our main result (Theorem 3.1) characterizes isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ as mappings which transfer apartments of $\mathcal{G}_k(V)$ to $J(n,k)$-subsets of $\mathcal{G}_{k'}(V')$. As a consequence, we get a generalization of the above mentioned result on apartments preserving mappings.

2. Grassmann graph and Johnson graph

2.1. **Graph theory.** In this subsection we recall some concepts of the general graph theory.

A subset in the vertex set of a graph is called a *clique* if any two distinct vertices in this subset are adjacent (connected by an edge). Every clique is contained in a maximal clique (this is trivial if the vertex set is finite and we use Zorn lemma in the infinite case).

The *distance* between two vertices in a connected graph Γ is defined as the smallest number i such that there is a path consisting of i edges and connecting these vertices. The *diameter* of Γ is the greatest distance between two vertices.

An *embedding* of a graph Γ in a graph Γ' is an injection of the vertex set of Γ to the vertex set of Γ' such that adjacent vertices go to adjacent vertices and non-adjacent vertices go to non-adjacent vertices. Every surjective embedding is an isomorphism. An embedding is said to be *isometric* if it preserves the distance between any two vertices. Every embedding preserves the distances 1 and 2. Thus any embedding of a graph with diameter 2 is isometric.

2.2. **Grassmann graph.** Let V be an n-dimensional left vector space over a division ring R. For every $k \in \{0, \dots, n\}$ we denote by $\mathcal{G}_k(V)$ the Grassmannian formed by k-dimensional subspaces of V. Then $\mathcal{G}_0(V) = \{0\}$ and $\mathcal{G}_n(V) = \{V\}$. In the case when $1 \leq k \leq n-1$, two elements of $\mathcal{G}_k(V)$ are said to be *adjacent* if their intersection is $(k-1)$-dimensional (this is equivalent to the fact that their sum is $(k+1)$-dimensional).

The *Grassmann graph* $\Gamma_k(V)$ is the graph whose vertex set is $\mathcal{G}_k(V)$ and whose edges are pairs of adjacent k-dimensional subspaces. The graph $\Gamma_k(V)$ is connected, the distance $d(S,U)$ between two vertices $S, U \in \mathcal{G}_k(V)$ is equal to

$$k - \dim(S \cap U) = \dim(S + U) - k$$

and the diameter of Γ_k is equal to $\min\{k, n-k\}$.

Let V^* be the dual vector space. This is an n-dimensional left vector space over the opposite division ring R^* (the division rings R and R^* have the same set of elements and the same additive operation, the multiplicative operation $*$ on R^* is defined by the formula $a * b := ba$ for all $a, b \in R$). The second dual space V^{**} is canonically isomorphic to V.

For a subset $X \subset V$ the subspace
$$X^0 := \{\ x^* \in V^*\ :\ x^*(x) = 0 \quad \forall\, x \in X\ \}$$
is called the *annihilator* of X. The dimension of X^0 is equal to the codimension of $\langle X \rangle$. The annihilator mapping of the set of all subspaces of V to the set of all subspaces of V^* is bijective and reverses the inclusion relation, i.e.
$$S \subset U \iff U^0 \subset S^0$$
for any subspaces $S, U \subset V$. Since $S^{00} = S$ for every subspace $S \subset V$, the inverse bijection is also the annihilator mapping. The restriction of the annihilator mapping to each $\mathcal{G}_k(V)$ is an isomorphism of $\Gamma_k(V)$ to $\Gamma_{n-k}(V^*)$.

Lemma 2.1. *If $S_1, \dots, S_m$ are subspaces of V then*
$$(S_1 + \dots + S_m)^0 = (S_1)^0 \cap \dots \cap (S_m)^0,$$
$$(S_1 \cap \dots \cap S_m)^0 = (S_1)^0 + \dots + (S_m)^0.$$

Consider incident subspaces $S \in \mathcal{G}_s(V)$ and $U \in \mathcal{G}_u(V)$ such that $s < k < u$. We define
$$[S, U]_k := \{\ P \in \mathcal{G}_k(V)\ :\ S \subset P \subset U\ \}.$$
In the case when $U = V$ or $S = 0$, this subset will be denoted by $[S\rangle_k$ or $\langle U]_k$, respectively. Subsets of such type are called *parabolic subspace* of $\mathcal{G}_k(V)$, see [11, Section 3.1].

There is the natural isometric embedding Φ_S^U of $\Gamma_{k-s}(U/S)$ in $\Gamma_k(V)$ which sends every $(k-s)$-dimensional subspace of U/S to the corresponding k-dimensional subspace of V. In the case when $U = V$ or $S = 0$, this embedding will be denoted by Φ_S or Φ^U, respectively. The image of Φ_S^U is the parabolic subspace $[S, U]_k$.

If $k = 1, n-1$ then any two distinct vertices of $\Gamma_k(V)$ are adjacent. In the case when $1 < k < n-1$, there are precisely the following two types of maximal cliques of $\Gamma_k(V)$:

- the *star* $[S\rangle_k$, $S \in \mathcal{G}_{k-1}(V)$,
- the *top* $\langle U]_k$, $U \in \mathcal{G}_{k+1}(V)$.

The annihilator mapping transfers every parabolic subspace $[S, U]_k$ to the parabolic subspace $[U^0, S^0]_{n-k}$; in particular, it sends stars to tops and tops to stars.

2.3. **Johnson graph.** The *Johnson graph* $J(n, k)$ is the graph whose vertices are k-element subsets of $\{1, \dots, n\}$ and whose edges are pairs of k-element subsets with $(k-1)$-element intersections. The graph $J(n, k)$ is connected, the distance $d(X, Y)$ between two vertices X, Y is equal to
$$k - |X \cap Y| = |X \cup Y| - k$$
and the diameter of $J(n, k)$ is equal to $\min\{k, n-k\}$. The mapping
$$X \to X^c := \{1, \dots, n\} \setminus X$$
is an isomorphism between $J(n, k)$ and $J(n, n-k)$.

If $k = 1, n-1$ then any two distinct vertices of $J(n, k)$ are adjacent. In the case when $1 < k < n-1$, there are precisely the following two types of maximal cliques of $J(n, k)$:

- the *star* which consists of all vertices containing a certain $(k-1)$-element subset,

- the *top* which consists of all vertices contained in a certain $(k+1)$-element subset.

The stars and tops of $J(n,k)$ consist of $n-k+1$ and $k+1$ vertices, respectively. The isomorphism $X \to X^c$ transfers stars to tops and tops to stars.

Let B be a base of V. The associated *apartment* of $\mathcal{G}_k(V)$ consists of all k-dimensional subspaces spanned by subsets of B. This is the image of an isometric embedding of $J(n,k)$ in $\Gamma_k(V)$. We will use the following facts:

- for any two k-dimensional subspaces of V there is an apartment of $\mathcal{G}_k(V)$ containing both of them;
- the annihilator mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_{n-k}(V^*)$ transfers apartments to apartments.

Let $[S,U]_k$ be a parabolic subspace of $\mathcal{G}_k(V)$. Let also B be a base of V such that S and U are spanned by subsets of B. The intersection of the corresponding apartment of $\mathcal{G}_k(V)$ with $[S,U]_k$ is said to be an *apartment* in the parabolic subspace $[S,U]_k$. This is the image of an isometric embedding of $J(u-s,k-s)$ in $\Gamma_k(V)$, where $s=\dim S$ and $u=\dim U$. The mapping Φ_S^U establishes a one-to-one correspondence between apartments of $\mathcal{G}_{k-s}(U/S)$ and apartments of the parabolic subspace $[S,U]_k$.

2.4. **Isometric embeddings of Johnson graphs in Grassmann graphs.** Let V' be an n'-dimensional left vector space over a division ring R'. Isometric embeddings of $J(n,k)$ in $\Gamma_{k'}(V')$ are classified in [12]. The existence of such embeddings implies that the diameter of $J(n,k)$ is not greater than the diameter of $\Gamma_{k'}(V')$, i.e.

$$\min\{k,n-k\} \le \min\{k',n'-k'\}. \tag{2.1}$$

Since $J(n,k)$ and $J(n,n-k)$ are isomorphic, we can suppose that $k \le n-k$. Then

$$k \le \min\{k',n-k,n'-k'\}.$$

The case $k=1$ is trivial: any two distinct vertices of $J(n,1)$ are adjacent and every isometric embedding of $J(n,1)$ in $\Gamma_{k'}(V')$ is a bijection to a clique of $\Gamma_{k'}(V')$.

We say that a subset $X \subset V$ is *m-independent* if every m-element subset of X is independent. If $x_1,\dots,x_m$ are linearly independent vectors of V and

$$x_{m+1} = a_1x_1 + \dots + a_mx_m,$$

where each a_i is non-zero, then $x_1,\dots,x_{m+1}$ form an m-independent subset. Every n-independent subset of V consisting of n vectors is a base of V. By [12, Proposition 1], if the division ring R is infinite then for every natural integer $l \ge n$ there is an n-independent subset of V consisting of l vectors.

Suppose that $k < n-k$ and X is a $(2k)$-independent subset of V consisting of l vectors. Every k-element subset of X spans a k-dimensional subspace and we denote by $\mathcal{J}_k(X)$ the set formed by all such subspaces. This is the image of an isometric embedding of $J(l,k)$ in $\Gamma_k(V)$. We will write $\mathcal{J}_k^*(X)$ for the subset of $\mathcal{G}_{n-k}(V^*)$ consisting of the annihilators of elements from $\mathcal{J}_k(X)$. If X is a base of V then $\mathcal{J}_k(X)$ and $\mathcal{J}_k^*(X)$ are apartments of $\mathcal{G}_k(V)$ and $\mathcal{G}_{n-k}(V^*)$, respectively.

The images of isometric embeddings of $J(n,k)$ in $\Gamma_{k'}(V')$ are called *$J(n,k)$-subsets* of $\mathcal{G}_{k'}(V')$.

Theorem 2.1 ([12]). *Let $\mathcal{J}$ be a $J(n,k)$-subset of $\mathcal{G}_{k'}(V')$ and $1<k\le n-k$. In the case when $n=2k$, there exist $S \in \mathcal{G}_{k'-k}(V')$ and $U \in \mathcal{G}_{k'+k}(V')$ such that $\mathcal{J}$ is*

an apartment in the parabolic subspace $[S,U]_{k'}$, i.e.

$$\mathcal{J}=\Phi^U_S(\mathcal{A}),$$

where $\mathcal{A}$ is an apartment of $\mathcal{G}_k(U/S)$. If $k<n-k$ then one of the following possibilities is realized:

1. *there exist $S\in\mathcal{G}_{k'-k}(V')$ and a $(2k)$-independent n-element subset $X\subset V'/S$ such that*
$$\mathcal{J}=\Phi_S(\mathcal{J}_k(X));$$
2. *there exist $U\in\mathcal{G}_{k'+k}(V')$ and a $(2k)$-independent n-element subset $Y\subset U^*$ such that*
$$\mathcal{J}=\Phi^U(\mathcal{J}^*_k(Y)).$$

In the case when $1<k<n-k$, we say that $\mathcal{J}$ is a $J(n,k)$-subset of *first* or of *second type* if the corresponding possibility is realized. The annihilator mapping changes types of $J(n,k)$-subsets.

Remark 2.1. Suppose that $1<k<n-k$ and $\mathcal{J}\subset\mathcal{G}_{k'}(V')$ is a $J(n,k)$-subset of second type. Let U and Y be as in Theorem 2.1. The annihilators of vectors belonging to Y form an n-element subset $\mathcal{Y}\subset\mathcal{G}_{k'+k-1}(U)$. Every element of $\mathcal{J}$ can be presented as the intersection of k distinct elements of $\mathcal{Y}$.

Let $\mathcal{C}$ be a maximal clique of $\Gamma_{k'}(V')$ (a star or a top). As above, we suppose that $\mathcal{J}$ is a $J(n,k)$-subset of $\mathcal{G}_{k'}(V')$ and $1<k\le n-k$. If $\mathcal{J}\cap\mathcal{C}$ contains more than one element than it is a maximal clique of the restriction of $\Gamma_{k'}(V')$ to $\mathcal{J}$ (this restriction is isomorphic to $J(n,k)$). In this case, we say that $\mathcal{J}\cap\mathcal{C}$ is a *star* or a *top* of $\mathcal{J}$ if $\mathcal{C}$ is a star or a top, respectively.

Lemma 2.2. *Suppose that $1<k<n-k$. If $\mathcal{J}$ is a $J(n,k)$-subset of first type then the stars and tops of $\mathcal{J}$ consist of $n-k+1$ and $k+1$ vertices, respectively. In the case when $\mathcal{J}$ is a $J(n,k)$-subset of second type, the stars and tops of $\mathcal{J}$ consist of $k+1$ and $n-k+1$ vertices, respectively.*

Proof. Easy verification. □

Lemma 2.2 shows that the two above determined classes of $J(n,k)$-subsets are disjoint.

2.5. **Isometric embeddings of Grassmann graphs.** Isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ are classified in [13]. As in the previous subsection, we have (2.1) which implies that the diameter of $\Gamma_k(V)$ is not greater than the diameter of $\Gamma_{k'}(V')$.

A mapping $l:V\to V'$ is called *semilinear* if

$$l(x+y)=l(x)+l(y)$$

for all $x,y\in V$ and there is a homomorphism $\sigma:R\to R'$ such that

$$l(ax)=\sigma(a)l(x)$$

for all $a\in R$ and $x\in V$. If l is non-zero then there is only one homomorphism σ satisfying this condition. Every non-zero homomorphism of R to R' is injective.

A semilinear injection of V to V' is said to be a *semilinear m-embedding* if it transfers any m linearly independent vectors to linearly independent vectors. The existence of such mappings implies that $m\le n'$. A semilinear n-embedding of V in V' will be called a *semilinear embedding*. It maps every independent subset to an

independent subset which means that $n \le n'$. For any natural integers $p \ge 3$ and q there is a semilinear p-embedding of a $(p+q)$-dimensional vector space which is not a $(p+1)$-embedding [8].

Let $l : V \to V'$ be a semilinear m-embedding. If P is a k-dimensional subspace of V and $k \le m$ then $\langle l(P) \rangle$ is a k-dimensional subspace of V'. So, for every $k \in \{1, \dots, m\}$ we have the mapping

$$(l)_k : \mathcal{G}_k(V) \to \mathcal{G}_k(V')$$
$$P \to \langle l(P) \rangle$$

and the mapping

$$(l)^*_k : \mathcal{G}_k(V) \to \mathcal{G}_{n'-k}(V'^*)$$
$$P \to \langle l(P) \rangle^0.$$

In the case when $k < m$, these mappings are injective and transfer adjacent subspaces to adjacent subspaces. If $2k \le m$ then $(l)_k$ and $(l)^*_k$ are isometric embeddings of $\Gamma_k(V)$ in $\Gamma_k(V')$ and $\Gamma_{n'-k}(V'^*)$, respectively.

Theorem 2.2 ([13])**.** *Let f be an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ and $1 < k \le n-k$. Then one of the following possibilities is realized:*

(1) *there exist $S \in \mathcal{G}_{k'-k}(V')$ and a semilinear $(2k)$-embedding $l : V \to V'/S$ such that $f = \Phi_S \circ (l)_k$;*
(2) *there exist $U \in \mathcal{G}_{k'+k}(V')$ and a semilinear $(2k)$-embedding $s : V \to U^*$ such that $f = \Phi^U \circ (s)^*_k$.*

In particular, if $n = 2k$ then there exist incident $S \in \mathcal{G}_{k'-k}(V')$ and $U \in \mathcal{G}_{k'+k}(V')$ such that f is induced by a semilinear embedding $l : V \to U/S$ or a semilinear embedding $s : V \to (U/S)^*$, i.e.

$$f = \Phi^U_S \circ (l)_k \quad \textit{or} \quad f = \Phi^U_S \circ (s)^*_k.$$

The case $k = 1, n-1$ is trivial. The case when $1 < k \le n-k$ is considered in Theorem 2.2. Suppose that $n-k < k < n-1$. Since $\Gamma_k(V)$ and $\Gamma_{n-k}(V^*)$ are canonically isomorphic, every isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ can be considered as an isometric embedding of $\Gamma_{n-k}(V^*)$ in $\Gamma_{k'}(V')$. The latter embedding is one of the mappings described in Theorem 2.2. In contrast to the case when $1 < k \le n-k$, we can not show that isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ are defined by semilinear mappings of V.

3. Main result

Let f be a mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_{k'}(V')$. If the restriction of f to every apartment of $\mathcal{G}_k(V)$ is an isometric embedding of $J(n,k)$ in $\Gamma_{k'}(V')$ then f is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$. This follows from the fact that for any two elements of $\mathcal{G}_k(V)$ there is an apartment containing both of them.

We say that f is a *J-mapping* if it sends every apartment of $\mathcal{G}_k(V)$ to a $J(n,k)$-subset. Every isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ satisfies this condition. Our main result states that this property characterizes isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$.

Theorem 3.1. *Every J-mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_{k'}(V')$ is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$.*

Some corollaries of Theorem 3.1 will be given in Section 6.

4. Intersections of $J(n,k)$-subsets

4.1. **Special subsets.** Let $X = \{x_1, \dots, x_n\}$ be a $(2k)$-independent subset of a vector space W (the dimension of W is assumed to be not less than $2k$ and $n \geq 2k$) and let $k \geq 2$. Consider the set $\mathcal{J} = \mathcal{J}_k(X)$ formed by all k-dimensional subspaces spanned by subsets of X. For every $i \in \{1, \dots, n\}$ we denote by $\mathcal{J}(+i)$ and $\mathcal{J}(-i)$ the sets consisting of all elements of $\mathcal{J}$ which contain x_i and do not contain x_i, respectively. Also, we write $\mathcal{J}(+i,+j)$ for the intersection of $\mathcal{J}(+i)$ and $\mathcal{J}(+j)$. Every

$$\mathcal{J}(+i,+j) \cup \mathcal{J}(-i), \quad i \neq j$$

is said to be a *special* subset of $\mathcal{J}$.

We say that a subset $\mathcal{X} \subset \mathcal{J}$ is *inexact* if there is a $(2k)$-independent n-element subset $Y \subset W$ such that $\mathcal{J}_k(Y) \neq \mathcal{J}$ (at least one of the vectors belonging to Y is not a scalar multiple of a vector from X) and $\mathcal{X} \subset \mathcal{J}_k(Y)$.

Lemma 4.1. *Every inexact subset is contained in a special subset.*

Proof. Let $\mathcal{X}$ be an inexact subset. Denote by S_i the intersection of all elements of $\mathcal{X}$ containing x_i and set $S_i = 0$ if there are no elements of $\mathcal{X}$ containing x_i. There is at least one i such that $S_i \neq \langle x_i \rangle$ (otherwise, $\mathcal{X}$ is not inexact). Then $S_i = 0$ or $\dim S_i \geq 2$. In the first case, $\mathcal{X}$ is contained in $\mathcal{J}(-i)$ which gives the claim. If $\dim S_i \geq 2$ then the inclusion

$$\mathcal{X} \subset \mathcal{J}(+i,+j) \cup \mathcal{J}(-i)$$

holds for any $j \neq i$ such that $x_j \in S_i$. □

Lemma 4.2. *If X is independent then the class of maximal inexact subsets coincides with the class of special subsets.*

Proof. By Lemma 4.1, it sufficient to show that every special subset is inexact. Since X is independent,

$$Y := (X \setminus \{x_i\}) \cup \{x_i + x_j\}$$

is independent and $\mathcal{J}_k(Y)$ contains the special subset $\mathcal{J}(+i,+j) \cup \mathcal{J}(-i)$. □

Remark 4.1. Suppose that $R = \mathbb{Z}_2$ and $X = \{x_1, \dots, x_5\}$, where $x_1, \dots, x_4$ are linearly independent vectors and

$$x_5 = x_1 + \dots + x_4.$$

Then $k = 2$ and X is 4-independent. The vectors $x_1 + x_2, x_3, x_4, x_5$ are not linearly independent and x_1 can not be replaced by $x_1 + x_2$ as in the proof of Lemma 4.2. The subspace $\langle x_1, x_2 \rangle$ contains only three non-zero vectors — $x_1, x_2, x_1 + x_2$. This means that $\mathcal{J}(+1,+2) \cup \mathcal{J}(-1)$ can not be inexact. The same arguments show that every special subset is not inexact.

Remark 4.2. It is not difficult to prove that all special subsets are inexact if R is infinite, but we do not need this fact.

The subsets $\mathcal{J}(+i,+j)$ and $\mathcal{J}(-i)$ are disjoint. This means that every special subset contains precisely

$$a(n,k) := |\mathcal{J}(+i,+j)| + |\mathcal{J}(-i)| = \binom{n-2}{k-2} + \binom{n-1}{k}$$

elements. Lemma 4.1 implies the following.

Lemma 4.3. *If an inexact subset consists of $a(n,k)$ elements then it is a special subset.*

A subset $\mathcal{X} \subset \mathcal{J}$ is said to be *complement* if $\mathcal{J} \setminus \mathcal{X}$ is special, i.e.

$$\mathcal{J} \setminus \mathcal{X} = \mathcal{J}(+i,+j) \cup \mathcal{J}(-i)$$

for some distinct i, j. Then

$$\mathcal{X} = \mathcal{J}(+i) \cap \mathcal{J}(-j).$$

This complement subset will be denoted by $\mathcal{J}(+i,-j)$.

Lemma 4.4. *Let $P, Q \in \mathcal{J}$. Then $d(P,Q) = m$ if and only if there are precisely*

$$(k-m)(n-k-m)$$

distinct complement subsets of $\mathcal{J}$ containing both P and Q.

Proof. The equality $d(P,Q) = m$ implies that

$$\dim(P \cap Q) = k-m \text{ and } \dim(P+Q) = k+m.$$

The complement subset $\mathcal{J}(+i,-j)$ contains both P and Q if and only if

$$x_i \in P \cap Q \text{ and } x_j \notin P+Q.$$

So, there are precisely $k-m$ possibilities for i and precisely $n-k-m$ possibilities for j. $\square$

4.2. **Connectedness of the apartment graph.** Suppose that $1 < k \le n-k$. If X is a base of V then $\mathcal{J}_k(X)$ is an apartment of $\mathcal{G}_k(V)$ and, by Lemma 4.2, the class of maximal inexact subsets coincides with the class of special subsets. Two apartments of $\mathcal{G}_k(V)$ are said to be *adjacent* if their intersection is a maximal inexact subset. Consider the graph A_k whose vertices are apartments of $\mathcal{G}_k(V)$ and whose edges are pairs of adjacent apartments.

Proposition 4.1. *The graph A_k is connected.*

Proof. Let B and B' be bases of V. The associated apartments of $\mathcal{G}_k(V)$ will be denoted by $\mathcal{A}$ and $\mathcal{A}'$, respectively. Suppose that $\mathcal{A} \neq \mathcal{A}'$ and show that these apartments can be connected in A_k.

First we consider the case when $|B \cap B'| = n-1$. Let

$$B = \{x_1, \dots, x_{n-1}, x_n\} \text{ and } B' = \{x_1, \dots, x_{n-1}, x'_n\}.$$

Since $\mathcal{A} \neq \mathcal{A}'$, the vector x'_n is a linear combination of x_n and some others $x_{i_1}, \dots, x_{i_m}$. Clearly, we can suppose that

$$x'_n = ax_n + \sum_{i=1}^{m} a_i x_i \text{ with } m \le n-1.$$

We prove the statement induction on m. If $m = 1$ then

$$\mathcal{A} \cap \mathcal{A}' = \mathcal{J}(+n,+1) \cup \mathcal{J}(-n)$$

is a maximal inexact subset and $\mathcal{A}, \mathcal{A}'$ are adjacent. Let $m \ge 2$. Denote by $\mathcal{A}''$ the apartment of $\mathcal{G}_k(V)$ associated with the base $x_1, \dots, x_{n-1}, x''_n$, where

$$x''_n := ax_n + \sum_{i=1}^{m-1} a_i x_i.$$

By inductive hypothesis, $\mathcal{A}$ and $\mathcal{A}''$ can be can be connected in A_k. The equality

$$x'_n = x''_n + a_m x_m$$

guarantees that $\mathcal{A}''$ and $\mathcal{A}'$ are adjacent. This implies the existence of a path connecting $\mathcal{A}$ with $\mathcal{A}'$.

Now consider the case when $|B \cap B'| = m < n-1$ (possible $m = 0$). Suppose that

$$B \setminus B' = \{x_1, \dots, x_{n-m}\} \text{ and } x' \in B' \setminus B.$$

For every $i \in \{1, \dots, n-m\}$ we define

$$S_i := \langle B \setminus \{x_i\} \rangle.$$

Since the intersection of all S_i coincides with $\langle B \cap B' \rangle$ and x' does not belong to $\langle B \cap B' \rangle$, there is at least one S_i which does not contain x'. Then

$$B_1 := (B \setminus \{x_i\}) \cup \{x'\}$$

is a base of V. Denote by $\mathcal{A}_1$ the associated apartment of $\mathcal{G}_k(V)$. It is clear that

$$|B \cap B_1| = n-1 \text{ and } |B_1 \cap B'| = m+1.$$

The apartment $\mathcal{A}_1$ coincides with $\mathcal{A}$ (if x' is a scalar multiple of x_i) or $\mathcal{A}$ and $\mathcal{A}_1$ are connected in A_k. Step by step we construct a sequence of bases

$$B = B_0, B_1, \dots, B_{n-m} = B'$$

such that $|B_{i-1} \cap B_i| = n-1$ for every $i \in \{1, \dots, n-m\}$. Let $\mathcal{A}_i$ be the apartment of $\mathcal{G}_k(V)$ associated with B_i. Then for every $i \in \{1, \dots, n-m\}$ we have $\mathcal{A}_{i-1} = \mathcal{A}_i$ or $\mathcal{A}_{i-1}$ and $\mathcal{A}_i$ are connected in A_k. This means that $\mathcal{A} = \mathcal{A}_0$ and $\mathcal{A}' = \mathcal{A}_{n-m}$ are connected in A_k. □

4.3. **Intersections of $J(n,k)$-subsets of different types.** In this subsection we suppose that W is a $(2k)$-dimensional vector space and $k \geq 2$. Let

$$X = \{x_1, \dots, x_n\} \text{ and } Y = \{y^*_1, \dots, y^*_n\}, \ \ n > 2k$$

be $(2k)$-independent subsets of W and W^*, respectively. Denote by U_i the annihilator of y^*_i. This is a $(2k-1)$-dimensional subspace of W. Suppose that the following conditions hold:

- every U_i is spanned by a subset of X,
- every $\langle x_i \rangle$ is the intersection of some U_j.

Since X is a $(2k)$-independent subset, every U_i is spanned by a $(2k-1)$-element subset $X_i \subset X$ and it does not contain any vector of $X \setminus X_i$. Similarly, Y is $(2k)$-independent and every x_i is contained in precisely $2k-1$ distinct U_j whose intersection coincides with $\langle x_i \rangle$.

We will investigate the intersection

$$\mathcal{Z} := \mathcal{J}_k(X) \cap \mathcal{J}^*_k(Y).$$

It is formed by all elements of $\mathcal{G}_k(W)$ which are spanned by subsets of X and can be presented as the intersections of k distinct U_j.

We define

$$b(n,k) := \frac{\binom{2k-1}{k} n}{k}.$$

Note that this integer is not necessarily natural.

Lemma 4.5. $|\mathcal{Z}| \leq b(n,k)$.

Proof. Denote by $\mathcal{Z}_i$ the set of all elements of $\mathcal{Z}$ containing x_i. There are precisely $2k-1$ distinct U_j containing x_i and every element of $\mathcal{Z}$ is the intersection of k distinct U_j. This means that $\mathcal{Z}_i$ contains not greater than $\binom{2k-1}{k}$ elements. Since every element of $\mathcal{Z}$ belongs to k distinct $\mathcal{Z}_i$, we have

$$|\mathcal{Z}| = \frac{|\mathcal{Z}_1| + \cdots + |\mathcal{Z}_n|}{k}$$

which implies the required inequality. □

Lemma 4.6. *$a(n,k) > b(n,k)$ except the case when $n = 5$ and $k = 2$.*

Proof. We have

$$a(n,2) = 1 + \frac{(n-1)(n-2)}{2} = \frac{n^2 - 3n + 4}{2} \quad \text{and} \quad b(n,2) = \frac{3n}{2}.$$

An easy verification shows that the equality $a(n,2) > b(n,2)$ does not hold only for $n = 5$.

From this moment we suppose that $k \geq 3$. Then

$$a(n,k) = \binom{n-2}{k-2} + \binom{n-1}{k} = \frac{(n-2)!}{(k-2)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} =$$
$$= \frac{(n-2)\dots(n-k+1)\cdot k(k-1)}{k!} + \frac{(n-1)\dots(n-k)}{k!} =$$
$$= [k(k-1) + (n-1)(n-k)]\frac{(n-2)\dots(n-k+1)}{k!}$$

and

$$b(n,k) = \frac{\binom{2k-1}{k}n}{k} = \frac{n(2k-1)!}{k!k!} = \frac{n(2k-1)\dots(k+1)}{k!} =$$
$$= [n(k+1)]\frac{(2k-1)\dots(k+2)}{k!}.$$

Since $n \geq 2k+1$ and $k \geq 3$,

$$(n-1)(n-k) + k(k-1) = (n-1)(n-k) + (k+1)(k-1) - (k-1) \geq$$
$$\geq (n-1)(k+1) + (k+1)(k-1) - (k-1) = (n+k-2)(k+1) - (k-1) \geq$$
$$\geq (n+1)(k+1) - (k-1) = n(k+1) + 2 > n(k+1).$$

So,

$$k(k-1) + (n-1)(n-k) > n(k+1). \tag{4.1}$$

Also, $n \geq 2k+1$ implies that

$$n-2 \geq 2k-1, \dots, n-k+1 \geq k+2$$

and we have

$$(n-2)\dots(n-k+1) \geq (2k-1)\dots(k+2). \tag{4.2}$$

The inequality

$$a(n,k) = [k(k-1) + (n-1)(n-k)]\frac{(n-2)\dots(n-k+1)}{k!} >$$
$$> [n(k+1)]\frac{(2k-1)\dots(k+2)}{k!} = b(n,k)$$

follows from (4.1) and (4.2). □

Lemma 4.7. *If $n = 5$ and $k = 2$ then $|\mathcal{Z}| \leq 5 < 7 = a(5,2)$.*

Proof. In the present case, $U_1, \dots, U_5$ are 3-dimensional, each x_i is contained in precisely 3 distinct U_j and every element of $\mathcal{Z}$ is the intersection of 2 distinct U_j. If every U_i contains not greater than 2 elements of $\mathcal{Z}$ then $|\mathcal{Z}| \le \frac{2 \cdot 5}{2} = 5$ (since every element of $\mathcal{Z}$ is contained in 2 distinct U_j).

Suppose that U_1 is spanned by x_1, x_2, x_3 and contains 3 elements of $\mathcal{Z}$. These are $\langle x_1, x_2 \rangle$, $\langle x_1, x_3 \rangle$, $\langle x_2, x_3 \rangle$. Suppose that these subspaces are the intersections of U_1 with U_2, U_3, U_4. Then each x_i, $i \in \{1,2,3\}$ is contained in 3 distinct U_j, $j \in \{1,2,3,4\}$. The subspace U_5 contains at least one of x_i, $i \in \{1,2,3\}$ and this x_i is contained in 4 distinct U_j, a contradiction.

The same arguments show that every U_i contains not greater than 2 elements of $\mathcal{Z}$ and we get the claim. $\square$

Joining all results of this subsection, we get the following.

Lemma 4.8. $|\mathcal{Z}| < a(n,k)$.

5. Proof of Theorem 3.1

Let f be a J-mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_{k'}(V')$.

Lemma 5.1. *The mapping f is injective.*

Proof. Let P, Q be distinct elements of $\mathcal{G}_k(V)$. We take an apartment $\mathcal{A} \subset \mathcal{G}_k(V)$ containing P and Q. Since $f(\mathcal{A})$ is a $J(n,k)$-subset, $\mathcal{A}$ and $f(\mathcal{A})$ have the same number of elements which implies that $f(P) \neq f(Q)$. $\square$

Consider the mapping f_* which transfers every $P \in \mathcal{G}_{n-k}(V^*)$ to $f(P^0)$. This is a J-mapping of $\mathcal{G}_{n-k}(V^*)$ to $\mathcal{G}_{k'}(V')$. It is clear that f is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ if and only if f_* is an isometric embedding of $\Gamma_{n-k}(V^*)$ in $\Gamma_{k'}(V')$. Therefore, it sufficient to prove Theorem 3.1 only in the case when $k \le n-k$.

Suppose that $k = 1$, i.e. f is a J-mapping of $\mathcal{G}_1(V)$ to $\mathcal{G}_{k'}(V')$. Any distinct $P, Q \in \mathcal{G}_1(V)$ are adjacent and there is an apartment $\mathcal{A} \subset \mathcal{G}_1(V)$ containing P, Q. Since $f(\mathcal{A})$ is a $J(n,1)$-subset, $f(P)$ and $f(Q)$ are adjacent vertices of $\Gamma_{k'}(V')$. Thus f is an isometric embedding of $\Gamma_1(V)$ in $\Gamma_{k'}(V')$.

From this moment we suppose that $2 \le k \le n-k$. By Subsection 2.4, we have

$$k \le \min\{k', n-k, n'-k'\}.$$

Lemma 5.2. *If $n = 2k$ then there exists $S \in \mathcal{G}_{k'-k}(V')$ such that the image of f is contained in $[S\rangle_{k'}$.*

Proof. Let $\mathcal{A}$ and $\mathcal{A}'$ be distinct apartments of $\mathcal{G}_k(V)$. Then $f(\mathcal{A})$ and $f(\mathcal{A}')$ are $J(n,k)$-subsets and, since $n = 2k$, Theorem 2.1 implies that

$$f(\mathcal{A}) = \Phi_S(\mathcal{J}_k(X)) \text{ and } f(\mathcal{A}') = \Phi_{S'}(\mathcal{J}_k(X')),$$

where $S, S' \in \mathcal{G}_{k'-k}(V')$ and X, X' are independent $(2k)$-element subsets of V'/S and V'/S', respectively. We need to show that $S = S'$.

By Proposition 4.1, it is sufficient to consider the case when $\mathcal{A}$ and $\mathcal{A}'$ are adjacent. Then

$$|f(\mathcal{A}) \cap f(\mathcal{A}')| = |\mathcal{A} \cap \mathcal{A}'| = a(2k,k)$$

and

$$\mathcal{X} := (\Phi_S)^{-1}(f(\mathcal{A}) \cap f(\mathcal{A}'))$$

is a subset of $\mathcal{J}_k(X)$ consisting of $a(2k,k)$ elements. Since $S+S'$ is contained in all elements of $f(\mathcal{A})\cap f(\mathcal{A}')$, every element of $\mathcal{X}$ contains $T:=(S+S')/S$. If $S\neq S'$ then $t=\dim T\geq 1$ and

$$|\mathcal{X}|\leq \binom{2k-t}{k-t}$$

which implies that

$$|\mathcal{X}|\leq \binom{2k-1}{k-1}=\frac{(2k-1)!}{(k-1)!k!}=\binom{2k-1}{k}<\binom{2k-1}{k}+\binom{2k-2}{k-2}=a(2k,k),$$

a contradiction. Thus $S=S'$. $\square$

Lemma 5.3. *Suppose that $k<n-k$. If f transfers an apartment $\mathcal{A}\subset\mathcal{G}_k(V)$ to a $J(n,k)$-subset of first type then the images of all apartments of $\mathcal{G}_k(V)$ are $J(n,k)$-subsets of first type and there exists $S\in\mathcal{G}_{k'-k}(V')$ such that the image of f is contained in $[S\rangle_{k'}$.*

Proof. By our hypothesis,

$$f(\mathcal{A})=\Phi_S(\mathcal{J}_k(X)),$$

where $S\in\mathcal{G}_{k'-k}(V')$ and X is a $(2k)$-independent subset of V'/S consisting of n vectors

$$\overline{x}_1=x_1+S,\dots,\overline{x}_n=x_n+S.$$

Denote by S_i the $(k'-k+1)$-dimensional subspace of V' corresponding to $\overline{x}_i$. Every element of $f(\mathcal{A})$ is the sum of k distinct S_j.

Let $\mathcal{A}'$ be an apartment of $\mathcal{G}_k(V)$ distinct from $\mathcal{A}$. We need to show that $f(\mathcal{A}')$ is a $J(n,k)$-subset of first type and is contained in $[S\rangle_{k'}$. By Proposition 4.1, it is sufficient to consider the case when $\mathcal{A}$ and $\mathcal{A}'$ are adjacent. As in the proof of the previous lemma,

$$\mathcal{X}:=(\Phi_S)^{-1}(f(\mathcal{A})\cap f(\mathcal{A}'))$$

is a subset of $\mathcal{J}_k(X)$ consisting of $a(n,k)$ elements. There are the following possibilities:

(1) $\mathcal{X}$ is contained in a special subset of $\mathcal{J}_k(X)$,
(2) there is no special subset of $\mathcal{J}_k(X)$ containing $\mathcal{X}$.

Case (1). Every special subset of $\mathcal{J}_k(X)$ consists of $a(n,k)=|\mathcal{X}|$ elements. This implies that $\mathcal{X}$ is a special subset of $\mathcal{J}_k(X)$. Suppose that

$$\mathcal{X}=\mathcal{J}(+i,+j)\cup\mathcal{J}(-i)$$

(see Subsection 4.1 for the notation). We take any $(k-1)$-dimensional subspace $T\subset V'/S$ spanned by a subset of X containing $\overline{x}_j$. Then

$$\mathcal{S}:=\mathcal{J}_k(X)\cap[T\rangle_k$$

is a star of $\mathcal{J}_k(X)$ contained in $\mathcal{X}$ (if $P\in\mathcal{S}$ contains $\overline{x}_i$ then it belongs to $\mathcal{J}(+i,+j)$ and $P\in\mathcal{S}$ is an element of $\mathcal{J}(-i)$ if it does not contain $\overline{x}_i$).

Consider $\Phi_S(\mathcal{S})$. This is a star of $f(\mathcal{A})$. By Lemma 2.2, this star consists of $n-k+1$ vertices (since $f(\mathcal{A})$ is a $J(n,k)$-subset of first type). Also, it is contained in $\Phi_S(\mathcal{X})\subset f(\mathcal{A}')$ and Lemma 2.2 guarantees that $f(\mathcal{A}')$ is a $J(n,k)$-subset of first type.

We take $P,Q\in\mathcal{X}$ such that $P\cap Q=0$. The intersection of $\Phi_S(P)$ and $\Phi_S(Q)$ coincides with S. Since $\Phi_S(P)$ and $\Phi_S(Q)$ both belong to $f(\mathcal{A}')$ and $f(\mathcal{A}')$ is

a $J(n,k)$-subset of first type, the associated $(k'-k)$-dimensional subspace of V' coincides with S and $f(\mathcal{A}')$ is contained in $[S\rangle_{k'}$.

Case (2). For every $i\in\{1,\dots,n\}$ the intersection of all elements of $\mathcal{X}$ containing $\overline{x}_i$ coincides with $\langle\overline{x}_i\rangle$ (otherwise, as in the proof of Lemma 4.1 we show that $\mathcal{X}$ is contained in a special subset of $\mathcal{J}_k(X)$ which is impossible). Then the intersection of all elements of

$$\Phi_S(\mathcal{X})=f(\mathcal{A})\cap f(\mathcal{A}')$$

containing S_i coincides with S_i. This implies that the intersection of all elements of $f(\mathcal{A})\cap f(\mathcal{A}')$ is S.

Therefore, if $f(\mathcal{A}')$ is a $J(n,k)$-subset of first type then the associated $(k'-k)$-dimensional subspace of V' coincides with S, i.e. $f(\mathcal{A}')$ is contained in $[S\rangle_{k'}$. Then $\mathcal{X}$ is an inexact subset of $\mathcal{J}_k(X)$. By Lemma 4.3, $\mathcal{X}$ is a special subset of $\mathcal{J}_k(X)$ which is impossible.

So, $f(\mathcal{A}')$ is a $J(n,k)$-subset of second type. Then

$$f(\mathcal{A}')=\Phi^U(\mathcal{J}^*_k(Y)),$$

where $U\in\mathcal{G}_{k'+k}(V')$ and Y is a $(2k)$-independent subset of U^* consisting of n vectors $y^*_1,\dots,y^*_n$. Denote by U_i the annihilator of y^*_i (in U). By Remark 2.1, every element of $f(\mathcal{A}')$ is the intersection of k distinct U_j.

The set

$$(\Phi^U)^{-1}(f(\mathcal{A})\cap f(\mathcal{A}')) \tag{5.1}$$

is contained in $\mathcal{J}^*_k(Y)$. Denote by $\mathcal{Y}$ the subset of $\mathcal{J}_k(Y)$ formed by the annihilators of all elements of (5.1). It consists of $a(n,k)$ elements. If $\mathcal{Y}$ is contained in a special subset of $\mathcal{J}_k(Y)$ then it coincides with this special subset. In this case, there is a star $\mathcal{S}\subset\mathcal{J}_k(Y)$ contained in $\mathcal{Y}$. Let $\mathcal{S}^0$ be the subset of $\mathcal{J}^*_k(Y)$ consisting of the annihilators of all elements of $\mathcal{S}$. Then $\Phi^U(\mathcal{S}^0)$ is a top of $f(\mathcal{A}')$ contained in $f(\mathcal{A})\cap f(\mathcal{A}')$. This contradicts Lemma 2.2, since $f(\mathcal{A})$ and $f(\mathcal{A}')$ are $J(n,k)$-subsets of different types.

Thus there is no special subset of $\mathcal{J}_k(Y)$ containing $\mathcal{Y}$. This means that for every $i\in\{1,\dots,n\}$ the intersection of all elements of $\mathcal{Y}$ containing y^*_i coincides with $\langle y^*_i\rangle$. By Lemma 2.1, U_i is the sum of the annihilators (in U) of these elements; hence it is the sum of some elements of $f(\mathcal{A})\cap f(\mathcal{A}')$. Since every element of $f(\mathcal{A})$ is the sum of k distinct S_j,

(*) every U_i is the sum of some S_j.

This implies that every U_i contains S (since S is contained in all S_i) and $f(\mathcal{A}')$ is a subset of $[S\rangle_{k'}$ (every element of $f(\mathcal{A}')$ is the intersection of k distinct U_j).

Since the intersection of all elements of $f(\mathcal{A})\cap f(\mathcal{A}')$ containing S_i coincides with S_i and every element of $f(\mathcal{A}')$ is the intersection of k distinct U_j,

(**) every S_i is the intersection of some U_j.

Then every S_i is contained in U and $f(\mathcal{A})$ is a subset of $\langle U]_{k'}$ (since every element of $f(\mathcal{A})$ is the sum of k distinct S_j).

So, $f(\mathcal{A})$ and $f(\mathcal{A}')$ both are contained in $[S,U]_{k'}$. The vector space $W:=U/S$ is $2k$-dimensional. It is clear that

$$f(\mathcal{A})=\Phi^U_S(\mathcal{J}_k(X))\ \text{ and }\ f(\mathcal{A}')=\Phi^U_S(\mathcal{J}^*_k(Y')),$$

where Y' is the $(2k)$-independent n-element subset of W^* induced by Y. The annihilators of the vectors belonging to Y' are U_i/S, $i \in \{1, \dots, n\}$. The facts (*) and (**) guarantee that X and Y' satisfy the conditions of Subsection 4.3:

- the annihilator of every element of Y' is spanned by a subset of X,
- every $\langle \overline{x}_i \rangle$ is the intersection of the annihilators of some elements from Y'.

By Subsection 4.3,

$$\mathcal{Z} := \mathcal{J}_k(X) \cap \mathcal{J}^*_k(Y')$$

contains less than $a(n, k)$ elements. This contradicts the fact that

$$\Phi^U_S(\mathcal{Z}) = f(\mathcal{A}) \cap f(\mathcal{A}')$$

consists of $a(n, k)$ elements. So, the case (2) is impossible. □

By Lemma 5.3, if $k < n - k$ then the images of all apartments of $\mathcal{G}_k(V)$ are $J(n, k)$-subsets of the same type.

Suppose that one of the following possibilities is realized:

- $n = 2k$,
- $k < n - k$ and the images of all apartments of $\mathcal{G}_k(V)$ are $J(n, k)$-subsets of first type.

By Lemmas 5.2 and 5.3, the image of f is contained in $[S\rangle_{k'}$ with $S \in \mathcal{G}_{k'-k}(V')$. This implies the existence of a mapping

$$g : \mathcal{G}_k(V) \to \mathcal{G}_k(V'/S)$$

such that $f = \Phi_S \circ g$. This is a J-mapping which transfers every apartment of $\mathcal{G}_k(V)$ to a certain $J_k(X)$, where X is a $(2k)$-independent n-element subset of V'/S. Using results of Subsection 4.1, we prove the following.

Lemma 5.4. *The mapping g is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(V'/S)$.*

Proof. Let $P, Q \in \mathcal{G}_k(V)$ and let $\mathcal{A}$ be an apartment of $\mathcal{G}_k(V)$ containing P and Q. If $\mathcal{X}$ is a special subset of $\mathcal{A}$ then $\mathcal{X} = \mathcal{A} \cap \mathcal{A}'$, where $\mathcal{A}'$ is an apartment of $\mathcal{G}_k(V)$ adjacent with $\mathcal{A}$. By Subsection 4.1,

$$g(\mathcal{X}) = g(\mathcal{A}) \cap g(\mathcal{A}')$$

is an inexact subset of $g(\mathcal{A})$. It consists of $a(n, k)$ elements and Lemma 4.3 implies that $g(\mathcal{X})$ is a special subset of $g(\mathcal{A})$. Since $\mathcal{A}$ and $g(\mathcal{A})$ have the same number of special subsets, a subset of $\mathcal{A}$ is special if and only if its image is a special subset of $g(\mathcal{A})$. Then $\mathcal{X}$ is a complement subset of $\mathcal{A}$ if and only if $g(\mathcal{X})$ is a complement subset of $g(\mathcal{A})$. Lemma 4.4 implies that

$$d(P, Q) = d(g(P), g(Q))$$

and we get the claim. □

Since Φ_S is an isometric embedding of $\Gamma_k(V'/S)$ in $\Gamma_{k'}(V')$, Lemma 5.4 guarantees that $f = \Phi_S \circ g$ is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$.

Now suppose that $k < n - k$ and the images of all apartments of $\mathcal{G}_k(V)$ are $J(n, k)$-subsets of second type. Consider the mapping f^* which sends every $P \in \mathcal{G}_k(V)$ to $f(P)^0$. This is a J-mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_{n'-k'}(V'^*)$. It transfers every apartment of $\mathcal{G}_k(V)$ to a $J(n, k)$-subset of first type. Then f^* is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{n'-k'}(V'^*)$ which means that f is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$.

6. Strong J-mappings

A J-mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_{k'}(V')$ is said to be *strong* if there is an apartment of $\mathcal{G}_k(V)$ whose image is an apartment in a parabolic subspace of $\mathcal{G}_{k'}(V')$. The apartments preserving mappings considered in [11, Section 3.4] are strong J-mappings.

If $n = 2k \geq 4$ then every J-mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_{k'}(V')$ is strong (Theorem 2.1) and, by Theorems 2.2 and 3.1, it is induced by a semilinear embedding of V in U/S or a semilinear embedding of V in $(U/S)^*$, where

$$S \in \mathcal{G}_{k'-k}(V') \text{ and } U \in \mathcal{G}_{k'+k}(V').$$

In this section, we show that all strong J-mappings of $\mathcal{G}_k(V)$ to $\mathcal{G}_{k'}(V')$ are induced by semilinear embeddings if $1 < k < n-1$. For $k = 1, n-1$ this fails [5].

First we prove the following generalization of [11, Theorem 3.10].

Corollary 6.1. *If $n = n'$ and $1 < k < n-1$ then every strong J-mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_k(V')$ is induced by a semilinear embedding of V in V' or a semilinear embedding of V in V'^* and the second possibility can be realized only in the case when $n = 2k$.*

Proof. Let f be a strong J-mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_k(V')$. By Theorem 3.1, f is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(V')$. We suppose that $n = n'$ and $1 < k < n-1$. Then there is an apartment $\mathcal{A} \subset \mathcal{G}_k(V)$ such that $f(\mathcal{A})$ is an apartment of $\mathcal{G}_k(V')$.

In the case when $n = 2k$, the required statement follows from Theorem 2.2.

If $k < n-k$ then, by Theorem 2.2, we have the following possibilities:

- $f = (l)_k$, where $l : V \to V'$ is a semilinear $(2k)$-embedding;
- $f = (s)^*_k$, where $s : V \to U^*$ is a semilinear $(2k)$-embedding and U is a $(2k)$-dimensional subspace of V'.

In the second case, the image of f is contained in $\langle U]_k$. Since $2k < n$, $\langle U]_k$ does not contain any apartment of $\mathcal{G}_k(V')$. So, this case is impossible and $f = (l)_k$. Then l transfers any base of V associated with $\mathcal{A}$ to a base of V'. This implies that l is a semilinear embedding.

Let $k > n-k$. Consider the mapping which transfers every $P \in \mathcal{G}_{n-k}(V^*)$ to $f(P^0)^0$. This is a strong J-mapping of $\mathcal{G}_{n-k}(V^*)$ to $\mathcal{G}_{n-k}(V'^*)$. By the arguments given above, it is induced by a semilinear embedding $s : V^* \to V'^*$. Denote by g the mapping of the set of all subspaces of V to the set of all subspaces of V' which sends every P to $s(P^0)^0$. By [11, Subsection 3.4.3], it is induced by a semilinear embedding $l : V \to V'$, i.e.

$$g(P) = \langle l(P) \rangle$$

for every subspace $P \subset V$. Since the restriction of g to $\mathcal{G}_k(V)$ coincides with f, we have $f = (l)_k$. □

Corollary 6.2. *Suppose that $1 < k < n-1$ and $n \neq 2k$. Then for every strong J-mapping $f : \mathcal{G}_k(V) \to \mathcal{G}_{k'}(V')$ one of the following possibilities is realized:*

1. *there exist $S \in \mathcal{G}_{k'-k}(V')$ and $U \in \mathcal{G}_{n+k'-k}(V')$ such that $f = \Phi^U_S \circ (l)_k$, where $l : V \to U/S$ is a semilinear embedding;*
2. *there exist $S' \in \mathcal{G}_{n'-k'-k}(V'^*)$ and $U' \in \mathcal{G}_{n+n'-k'-k}(V'^*)$ such that $f = \mathrm{A} \circ \Phi^{U'}_{S'} \circ (l)_k$, where $l : V \to U'/S'$ is a semilinear embedding and A is the annihilator mapping of $\mathcal{G}_{n'-k'}(V'^*)$ to $\mathcal{G}_{k'}(V')$.*

Proof. By Theorem 3.1, f is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$. Suppose that $k < n-k$. Theorem 2.2 states that one of the following possibilities is realized:

- $f = \Phi_S \circ (l)_k$, where $S \in \mathcal{G}_{k'-k}(V')$ and $l : V \to V'/S$ is semilinear $(2k)$-embedding;
- $f = \Phi^U \circ (s)^*_k$, where $U \in \mathcal{G}_{k'+k}(V')$ and $s : V \to U^*$ is a semilinear $(2k)$-embedding.

As in Corollary 6.1, we establish that l and s both are semilinear embeddings.

We get a mapping of type (1) in the first case.

In the second case, the image of f is contained in $[T, U]_{k'}$, where $T \in \mathcal{G}_{k+k'-n}(V')$ is the annihilator of $s(V)$ in U. Consider the mapping f^* sending every $P \in \mathcal{G}_k(V)$ to $f(P)^0$. The image of this mapping is contained in $[S', U']_{n'-k'}$ with

$$S' := U^0 \in \mathcal{G}_{n'-k'-k}(V'^*) \text{ and } U' := T^0 \in \mathcal{G}_{n+n'-k'-k}(V'^*).$$

Then $f^* = \Phi^{U'}_{S'} \circ g$, where g is a J-mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_k(U'/S')$. This J-mapping is strong (since f and f^* are strong J-mappings). The dimension of U'/S' is equal to n and Corollary 6.1 implies that g is induced by a semilinear embedding of V in U'/S'. Thus f is a mapping of type (2).

Now suppose that $k > n - k$. The image of f coincides with the image of the mapping f_* which transfers every $P \in \mathcal{G}_{n-k}(V^*)$ to $f(P^0)$. This image is contained in

$$[N, M]_{k'}, \quad N \in \mathcal{G}_{k'-n+k}(V'), \quad M \in \mathcal{G}_{k'+k}(V')$$

(f_* is a mapping of type (1)) or it is a subset of

$$[S, U]_{k'}, \quad S \in \mathcal{G}_{k'-k}(V'), \quad U \in \mathcal{G}_{k'+n-k}(V')$$

(f_* is a mapping of type (2)).

In the second case, $f = \Phi^U_S \circ g$, where g is a strong J-mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_k(U/S)$. Since U/S is n-dimensional, Corollary 6.1 implies that g is induced by a semilinear embedding of V in U/S and f is a mapping of type (1).

Suppose that the image of f is contained in $[N, M]_{k'}$. As above, we consider the mapping f^* which sends every $P \in \mathcal{G}_k(V)$ to $f(P)^0$. Its image is a subset of $[S', U']_{n'-k'}$ with

$$S' := M^0 \in \mathcal{G}_{n'-k'-k}(V'^*) \text{ and } U' := N^0 \in \mathcal{G}_{n+n'-k'-k}(V'^*).$$

Then $f^* = \Phi^{U'}_{S'} \circ g$, where g is a strong J-mapping of $\mathcal{G}_k(V)$ to $\mathcal{G}_k(U'/S')$. The standard arguments show that f is a mapping of type (2). □

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